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Mathematics in Cognitive Science

Abstract

What role does mathematics play in cognitive science today, what role should mathematics play in cognitive science tomorrow? The cautious short answers are: to the factual question, a rather modest role, except in peripheral areas; to the normative question, a far greater role, as the periphery's place is reevaluated and as both cognitive science and mathematics grow. This paper aims at providing more detailed, perhaps more contentious answers.

1. Clearing the Ground: Mathematics, Models, and Cognitive Science

Cognitive science and mathematics do not relate to one another as two well-defined, stable entities: they evolve and in fact co-evolve. This of course happens whenever a new science starts looking for help from mathematics. Take physics, or economics: in both of these cases, mathematics has profoundly shaped the emerging science, and reciprocally the science has impacted mathematics by making it develop some specific tools (which then become part of a new branch which can be used elsewhere).[1] But cognitive science resembles biology more than these other disciplines: at least up until recently, mathematics was not seen by a majority of cognitive scientists as having an important role to play in their field. Unlike biology however, cognitive science is hardly a mature discipline, in fact it is more of a loose federation of research programs, still searching for unifying principles.

The very fact that mathematics has historically been peripheral to cognitive science, and cognitive science to mathematics, makes it imperative not to assume that the interaction must involve 'core' areas of both field. Logic was never, and arguably still is not a core area of mathematics, yet it was for a long time, and it remains to a large extent, suitably extended, the main representative of mathematics within cognitive science. Symmetrically, vision and motor control are not core areas of cognitive science, nor is the physiology of the single neuron or of cortical columns, yet they are the main recipients of knowledge stemming from such

1 This converse influence is of course much greater in the case of physics than in the case of economics: a large part of mathematics owes its existence to the requirements of physics, while the branches of mathematics which were developed in response to the specific needs of economics are few.

core areas of mathematics as functional analysis, topology, dynamical systems, group theory or probability. We should therefore keep an open mind as to what belongs to mathematics or cognitive science. Regarding the former, we should not rule out of bounds areas that at present lay at the periphery (say, logic, graph theory, computational geometry or theoretical computer science). Regarding the latter, we should refrain from imposing upon it some preconceived structure, with (cognitive) neuroscience, or artificial intelligence, or developmental psychology, or generative linguistics at its center, and, for example basic neuroscience, computational linguistics, artificial intelligence or motor coordination in subordinate positions. Cognitive science is forever reconfiguring and does not seem any closer to unification than when it emerged some 60 years ago. Only its nominal object, loosely defined as the conjunction of the mind and the brain, has remained fixed, with an increased emphasis, in the last couple of decades, on the context provided by the body.

Now that I have somewhat narrowed down the relata, I should say something about the relation(s) to be examined.

First, among the applications of mathematics to cognitive science, we need to distinguish those that merely (though perhaps importantly) impact one of the component disciplines or sub-disciplines from those that directly impact, or claim to impact, or may impact the enterprise as a whole, in its general methodology.

A second useful distinction we may wish to make is the following. Among the mathematical tools and techniques deployed in the various areas of cognitive science, some are of such general scope as to be equally applicable to areas unconnected to cognitive science: for example, statistical methods for the aggregation and assessment of experimental data that are extensively used in developmental psychology, in linguistics, in neuroscience, in neuropsychology, etc., but have no relevance to these areas *qua* members of the cognitive science federation: they serve the same purpose as they do in any one of the so-called special sciences. On the other hand, certain mathematical tools seem to have a significant impact on the content, or the conceptual structure, of the discipline that deploys them. The distinction is not necessarily sharp: the mathematics of neuroimaging, for example, although quite general – it works for medical imagery and many other kinds of imagery – significantly impacts cognitive neuroscience and in particular raises specific methodological problems. It has also been argued by Gigerenzer that the tools we use sometimes evolve into a structural principle or a general heuristic for the field.[2] Nonetheless, as a first approximation, it is both useful and feasible to concentrate on the second sort of mathematical application.

Third, we can ask whether mathematical modeling – the production of mathematical models of cognitive phenomena – exhausts the topic at hand, or whether mathematics can relate to cognitive science in a different way. The thought would

2 Gerd Gigerenzer, "From Tools to Theories: A Heuristic of Discovery in Cognitive Psychology", in: *Psychological Review* 98, 2, 1991, pp. 254–267.

be that mathematics provides, could or should provide, a *framework* for cognitive science; which would then be, or become, a fully mathematized science, in the way of physics for example. If we take our lead from the "queen of science", taken at the most elementary level of sophistication, we can for example distinguish between, on the one hand, calculus as a mathematical method (whose centrality need not be stressed), and on the other, the differential equations of the Earth-Moon system, or of the tides, or of the propagation of heat in a metallic bar: these are mathematical models of physical phenomena. The two are obviously related, and no less obviously distinct. Perhaps a helpful metaphor might be that calculus is, or is part of the language of physics, while models are descriptions or representations couched in that language. Alternatively, we could perhaps say that mathematics is constitutive of physics, as we know it today, with the consequence that a model in physics is almost by definition a mathematical object; while mathematics is not constitutive, e.g., of biology, whose models (moreover) are infrequently mathematical objects. Yet another way in which the difference is made manifest is that in physics, models are theories; in cognitive science (as in biology), models (in most approaches explored today) never rise to the status of theories – the one major exception being the proposal by classical AI to regard a computer program as a theory.[3]

Finally, we might want to set up a continuum between two polar situations. On one end, we would find "deep" mathematics (mathematical theories with conceptual depth, wide scope, powerful techniques) imparting intelligibility on some deep questions in cognitive science. On the other end, we would find simple mathematics used to describe or systematize fairly limited domains.

These four distinctions, though in large part conceptually independent, rather naturally give rise to two clusters of properties, characteristic of two opposing stances. The mathematically modest perspective is content with viewing mathematics as a toolbox providing methods, and material, some quite general, some more domain-specific, for opportunistically constructing models, piecemeal, of various cognitive phenomena at various levels of description. The mathematically ambitious perspective aims at couching cognition, so to speak, in the language of mathematics, and thereby revealing the deep structure of the mental realm, in which the piecemeal models of specific functions obtained from detailed empirical work would be seen to find their natural place.

3 See, e.g., Herbert Simon, "Artificial Intelligence: An Empirical Science", in: *Artificial Intelligence* 77, 1995, p. 97: "The theory is no more separable from the program than classical mechanics is from the mathematics of the laws of motion".

2. From Prehistorical to Postmodern Cognitive Science: Five Stages

As is well known, cognitive science has undergone a number of stages, since its inception, which can be placed in the 1940s. It is important to have this history in mind, in schematic form, for to each stage corresponds a specific framework for the mathematics of cognitive science. The following thumbnail descriptions are provided as no more than an aide-mémoire.

The prehistorical phase (1942–1956) was centered on the recently reborn logic and the just emerging cybernetics.[4] Logic was developed as a branch of mathematics and as a language for representing certain essential mental operations. It was mechanicized in the hands of Turing[5] and others, and biologized by McCulloch and Pitts and others.[6] The broad ambition of cybernetics was to provide an overarching theory of mind, brain and machines, couched in the appropriate language of information and control. It included a branch concerned with higher functions, with logic as its main tool, and a branch concerned with perception and motricity, with some classical and new mathematics distinct from logic.

The first phase of the historical period (roughly 1956–1980) centered on artificial intelligence (AI), broadly understood as the science of "intelligent" information processing, leading up to the so-called classical, or symbolic paradigm in cognitive science.[7] The formal systems of logic provided the language, and theories (at least notionally) took the form of (computer) programs; we would be more comfortable today calling them *models*, but at the time it was important not to let the theoretical ambition of AI be watered down: AI was to be the scientific theory of human intelligence (of cognition), not a mere methodology for producing intelligence-like effects. The needed mathematics was logic, automata theory, and the nascent computer science or informatics. However, a large part of the work was carried out with no visible help from the theoretical parts of these formal disciplines. In fact the deepest contributions concerned the development of program-

4 Cybernetics may in fact be regarded as an early form of cognitive science. See Jean-Pierre Dupuy, *On the Origins of Cognitive Science: The Mechanization of the Mind*. Cambridge, MA: MIT Press 2009; Steve Joshua Heims, *The Cybernetics Group*. Cambridge MA: MIT Press 1991.

5 Alan M. Turing, "On Computable Numbers, with an Application to the *Entscheidungsproblem*", in: *Proceedings of the London Mathematical Society*, 42, 2, 1937, pp. 230-265; reprinted in: Martin Davis (Ed.), *The Undecidable*. Hewlett, NY: Raven Press 1965 and many other collections.

6 Warren S. McCulloch/Walter A. Pitts, "A Logical Calculus of Ideas Immanent in Nervous Activity", in: *Bulletin of Mathematical Biophysics* 5, 1943, pp. 115-133; reprinted in: Warren S. McCulloch, *Embodiments of Mind.* Cambridge, MA: MIT Press 1965; also in: James A. Anderson and Edward Rosenfeld (Eds.), *Neurocomputing. Foundations of Research.* Cambridge, MA: MIT Press 1988.

7 See e.g. Max Lungarella, Famiya Iida, Josh Bongard and Rolf Pfeifer (Eds.), *50 Years of Artificial Intelligence.* Berlin-Heidelberg: Springer 2007.

ming languages, first and foremost Lisp, then Prolog and more recently object-oriented languages such as Java, which made writing code for cognitive functions feasible. Actually producing a computer program for, say, chess or checkers playing, or scene recognition, or parsing, or writing a large corporation's paychecks, consisted in armchair construction of information-flow diagrams, an activity that can hardly be taken as part of mathematics. Exception must be made for the study of perceptual and motor functions, which recruited several high-powered mathematical areas, ranging from differential geometry to Fourier analysis and probability theory, and in fact extending them to meet specific requirements.

Next came (*ca.* 1980–1995) connectionism or the neural nets approach, which took up the perceptual strand of cybernetics and extended it into a full-fledged framework for cognitive science (and AI), competing with the classical, symbolic approach.[8] Connectionism, which comprises several rather distinct currents, can be applied at the functional or mental level, at the neuronal level, or again at an intermediate level, abstracted from the neuronal level and reflecting the "micro-structure" of cognition, understood in informational terms. The mathematics is here much more visible than in the symbolic approach, and also much richer and more varied, comprising fragments of linear algebra, of probability and signal theory, of analysis, and of dynamical systems, although seldom reaching great heights of sophistication.

The modern phase, to which the present still belongs, but is morphing into what I venture to call post-modern, is characterized, first and foremost, by the appearance of a new contender for the status of admiral discipline: cognitive neuroscience, supported by functional neuro-imaging technology but also by the strengthening of theoretical neuroscience, which consists in applying the methods of physical modeling to phenomena arising at various levels of organization of the nervous tissue.[9] Mathematical tools have become considerably more sophisticated. Functional imagery calls upon highly complex statistical methods aiming at providing a pictorial representation of the distributed activity in neuronal population, taking a gigantic mass of indirect signals as the basis of an inference to their sources. Theoretical neuroscience helps itself to a vast repertory of mathematical theories. The second most important feature of the modern phase is the return of the body, which appears not only under the guise of the brain, material "siege" of cognition, but also as organism and genuine bearer of cognition. With the body come perception and motricity, which, as we just saw, were never totally

8 See e.g. James A. Anderson, Andras Pellionisz and Edward Rosenfeld (Eds.), *Neurocomputing II.* Cambridge, MA: MIT Press 1990.

9 See e.g. Peter Dayan and Laurence F. Abott, *Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems*. Cambridge, MA: MIT Press 2001; Michael A. Arbib, *The Handbook of Brain Theory and Neural Networks*, 2nd ed. Cambridge, MA: MIT Press 2003; Michael Gazzaniga, *The Cognitive Neurosciences*, 4th ed., Cambridge, MA: MIT Press 2009.

neglected, but now take on an entirely novel dimension and call for pretty deep mathematics.

Post-modernism (a notion which I venture to propose here, but which to my knowledge has not been proposed under this or any other name by observers of contemporary cognitive science) is characterized by a breakdown of pragmatic unity and doctrinal consensus. Cognitive science is at a tipping point. Is it on the verge of disintegration, with a majority of programs recategorized inside neuroscience (and more broadly biology), and the rest reintegrating other main disciplines, or is it headed towards a fully integrated field, awaiting a new framework in which mathematics is likely to play a fundamental part? While this "crisis" is playing out, though, the working scientists are going into high gear, bringing to bear, in some areas, extremely powerful and wide-scope principles with strong mathematical content. The field is increasingly divided between areas in which a hard-science culture is required, and those in which it isn't, reconstructing, perhaps, the boundary between the natural and the human sciences.

3. Mathematics as a Local Player: a Sample

Not only do the stages overlap, so that the mathematical methods characteristic of the various phases actually have co-existed and were sometimes combined, but they abstract away from the divisions, at all times, within cognitive science. All along, besides the general framework to which some programs explicitly refer, specialties have existed and evolved quite independently, developing a proprietary methodology with owed little to the general framework. Typically, vision science, as has been already mentioned, though in a sense central to the project, due in particular to its uncharacteristic success, and yet in another peripheral, due to the its de-emphasis on *human* as opposed to *machine* (robotic) or *animal* vision, was off to an early start and not only exploited existing theories from contemporary mathematics, but developed its own mathematical tools. Meanwhile many other branches of cognitive science developed without any mathematics at all. Still, the overall trend has been a growing role of mathematics in the field.

As a second pass then, I offer a quick tour of a number of research programs, some of which were mentioned in passing, and which call on a variety of mathematical techniques or styles, as both language and modeling methodology (as per the distinction sketched in §1).

What follows is a mere sample, by no means an exhaustive list. I have divided it in three parts, which are not clear-cut but rather denote different attitudes and practices in the deployment of mathematics combined with differing approaches to cognition.

a. Abstract or pure information-processing theories (also known as computational theories)

i. Logic(s). The study of reasoning is probably the best-known subprogram of early cognitive science, a natural extension of the tradition of logic, taking on board two crucial dimensions, control and computational (neo-mechanical) feasibility, and straddling cognitive psychology and AI. In the widest sense, it can be argued that the guiding assumption of the early period of the field was that cognition is, at base, reasoning. Though this assumption is no longer in favor, reasoning, widely construed, remains a central topic. It encompasses not only deduction from firm premises in eternal propositional format (such things as “2+2 = 4” or “force equals mass times acceleration”), but also a large variety of inferential regimes (inductive, abductive …) deployed on different materials (non-purely propositional, non-eternal, non-firm, etc.), including coherence maintenance and belief revision, as well as problem-solving and even scientific inquiry. The development of non-standard systems of logic, including defeasible or non-monotonic logics, of algorithmic control systems, and of algorithmic complexity theory, clearly demonstrates the co-evolutionary process affecting cognitive science and mathematics. However, the part of cognitive science directly affected by the more sophisticated mathematical logic involved belongs to AI and computer science, rather than cognitive or developmental psychology or even formal theories of rationality, core areas which recruit no more than primitive mathematical techniques.

ii. Signal detection theory [SDT]. How to discriminate noise from signal, say in a visual or auditory scene, can be seen as a decision process. Probability theory is thus brought to bear on psychophysics, the study of perceptual systems as physical measurement devices. But SDT extends to a wider set of phenomena, including some that are more clearly cognitive, such as individual or collective decision-making under uncertainty.

iii. Control theory. Classical robotics relies heavily on control theory, a part of dynamical systems theory that also applies to the study of various biological processes. Here again, the initial target of the (fairly deep) mathematics involved leans towards the machine dimension of cognitive science, or the motoric dimension, long deemed somewhat peripheral. However, on the one hand this motoric dimension has recently been recognized as more important and more closely linked to cognition than was previously thought, and on the other, dynamical systems are propounded as an alternative to classical computational models for cognitive science at large.

iv. Machine learning. The first attempt at developing an information-theoretic approach to learning was initiated in the 1960s with the aim of formalizing induction in general, and more particularly, the inductive identification of the ambient language by the non-linguistic infant. The acquisition, by a child, of the grammar (syntax) of her mother tongue can be seen, and formalized, as a problem of induction: the innate language faculty provides a set of constraints which limit the set of possible grammars. The child’s job is to identify with which of the possible languages she is in fact confronted, on the limited basis of what she hears. This thought has led to the development of formal learning theory, which draws on fairly simple notions from discrete mathematics and recursive functions.[10] It has been extended to the study of scientific inquiry, seen as a process of induction from basic empirical data.[11]

A very different approach to machine learning, now generally preferred, is the PAC paradigm (probably approximate learning) developed in the early 1980s:[12] from a sample of the set to be "learned", PAC learning produces, with high probability, a generalization function which suitably approximates the given set. PAC involves sophisticated tools drawn from or developed within computational complexity theory.

v. Probability theory. Probability lies of course at the foundation of decision theory, an area that is traditionally claimed by economics as its core theory but is increasingly taken over by "neuro-economics", a joint venture of economics and cognitive science. Less known perhaps outside the field, but quite important, is the attempt to attack a very broad collection of cognitive processes[13] by postulating an optimizing principle operating on non-conscious sensations or data. The so-called Bayesian approach is now pre-eminent in vision science; it is also applied to the study of memory, and mobilizes fairly sophisticated mathematical tools.

vi. Game theory. Decision theory has gone collective with the help of game theory, specifically invented for that purpose. But again it is not widely known that game theory has become an instrument of choice for the evolutionary approaches of collective behavior, and is thus relevant for the study of social cognition, e.g. the natural basis of other-oriented behavior and norms.[14] The mathematical results required for the latter topic are however quite rudimentary.

vii. Category theory. Classical first-order logic has been pressed into service in the quest for formal models of natural language–this is the well-known program of Montague semantics. But just as category theory has claimed to provide mathematics with a better foundation than set theory, it has also been promoted as the best framework for the semantics of natural language and the associated field of categorization. Indeed, some authors have argued that category theory should

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10 Sanjay Jain, Daniel N. Osherson, James S. Royer and Arun Sharma, *Systems That Learn: An Introduction to Learning Theory (Learning, Development, and Conceptual Change)*, 2nd ed., Cambridge, MA: MIT Press 1999.

11 Eric Martin and Daniel N. Osherson, *Elements of Scientific Inquiry.* Cambridge, MA: MIT Press 1998.

12 Leslie Valiant, "A Theory of the Learnable", in: *Communications of the ACM* 27, 11, 1984, pp.1134-1142.

13 And even cognition as a whole; see Nick Chater and Mike Oaksford (Eds.), *The Probabilistic Mind: Prospects for Bayesian Cognitive Science*. New York: Oxford University Press 2008.

14 Robert Axelrod, *The Evolution of Cooperation*, Revised Edition. New York: Basic Books 2006.

replace logic (which in some sense it generalizes) as the *organon* of cognition.[15] Not surprisingly, this last example leads us back to the first as far as conceptual ambition goes: like logic in the early days, category theory is poised not (only) as a (modeling) tool for this or that cognitive process, but as the true language of cognition. It must be remarked that this is a minority view, ignored by a vast majority of scientists and philosophers of cognitive science.

b. Neural dynamics. In this group belong some applications of core mathematical theories to systems that take their inspiration from a general view of the basic structure of the brain.

i. Linear algebra and statistical physics are the indispensable tools to study the dynamics and learning capabilities of feed-forward layered networks of threshold automata, which constitute a large and well-studied family of neural nets. In the "parallel distributed processing" (PDP) view,[16] such systems are capable of supporting a wide variety of cognitive functions and are regarded as a basic architecture competing with the von Neumann computer. Mathematical analysis is essential in order to determine the conditions under which a system will stabilize, hence provide a definite output in response to a given input and even more importantly in order to define learning algorithms that work, for example retropropagation. Learning is of the essence for PDP as it allows a network to implement a given input-output function by being exposed to a set of examples.

ii. Statistical physics is also used, but at a deeper level, in the study of another family of neural nets, those that are fully interconnected (as opposed to feed-forward) and can thus be regarded as autonomous dynamical systems (they are sometimes called "attractor neural networks" or ANN).[17] The first example of this approach was proposed by physicist John Hopfield who exported a modeling technique perfected by solid-state physicists, the Ising model, to the study of a neural net that he could interpret as a device with a content-adressable memory.[18]

iii. Tools from advanced analysis (ordinary non-linear differential equations, partial differential equations, Fourier analysis and wavelets …) and from dynami-

15 François Magnan and Gonzalo E. Reyes, "Category Theory as a Conceptual Tool in the Study of Cognition", in: John Macnamara and Gonzalo E. Reyes (Eds.), *The Logical Foundations of Cognition.* New York Oxford: Oxford University Press 1994; Jaime Gómez and Ricardo Sanz, "Modeling Cognitive Systems with Category Theory. Towards Rigor in Cognitive Sciences", Tech. Report Universidad Politécnica de Madrid 2009.

16 David E. Rumelhart and James L. McClelland, eds., *Parallel Distributed Processing: Explorations in the Microstructure of Cognition*. Cambridge, MA: MIT Press 1986.

17 Daniel J. Amit, *Modeling Brain Function: The World of Attractor Neural Networks*. Cambridge: Cambridge University Press 1992.

18 John Hopfield, "Neural Networks and Physical Systems with Emergent Collective Computational Abilities", in: *Proceedings of the National Academy of Sciences USA* 79, 1982, pp. 2554-2558.

cal systems theory, often combined with stochastic methods, are deployed in several areas:[19]

— in basic (i.e. neuron-level) neuroscience, the propagation of the nerve impulse; the receptor field of the ganglia cells of the retina, which play a crucial role in early visual processing,[20] or of the simple cells in V1, which help the visual system deal with noisy stimuli; the functional architecture of V1, etc.[21]

— at a higher level of integration, the theory of networks of weakly coupled oscillators provides models for complex representations in the brain (i.e. representations that include independently identifiable components);

— at a functional level, abstracting away from neural implementation, image processing by biologically plausible or by artificial systems calls on variational methods.

iv. In the theory of motor control (balance, gait, reach, grasp, navigation …), geometry is increasingly regarded as encoded at a deep level in the relevant brain areas, which literally solve complex geometrical problems. Differential geometry and kinematics are thus seen not only as descriptive tools, but as naturally realized faculties of the brain.[22] On the side of artificial systems, robotics has directly or indirectly attracted the interest of top mathematicians working in such fields as algebraic and differential geometry, Lie theory, optimization theory as far back as the 19th century. Nowadays, robotics calls on a large spectrum of powerful theories, ranging from dynamical systems to Bayesian statistics, discrete and computational geometry or topology.[23]

4. Deep *vs* Shallow Engagement

I have up until now more or less explicitly indexed the depth of mathematization of a research program on the depth of the mathematics deployed. On that count, the mathematization of classical or symbolic AI or cognitive science is considerably shallower, than that of connectionism, and species of connectionism range from relatively less deep to quite deep; or again, formal learning theory is mathematically shallower than PAC learning. Similarly, theories of specific functions, such as language acquisition, phonology, pattern recognition, vision, motor con-

19 Alain Berthoz, "Rapport sur les liens entre mathématiques et neurosciences*»,* in: *Rapports sur la science et la technologie* 20, 2005, pp. 175-211.

20 David Marr, *Vision*. San Francisco: W.H.Freeman & Co Ltd 1982.

21 Jean Petitot, *Cognitive Morphodynamics: Dynamical Morphological Models of Constituency in Perception and Syntax*. New York: Peter Lang Pub Inc 2011.

22 Nikolai Bernstein, *The Coordination and Regulation of Movement.* New York: Pergamon Press 1967; Alain Berthoz, *The Brain's Sense of Movement.* Harvard: Harvard University Press 2002.

23 "The Interplay between Mathematics and Robotics", Summary of a Workshop, National Science Foundation, Arlington, VA, 15-17 May, 2000.

trol etc. can be roughly ordered according to the sophistication of the mathematics they use. This criterion also can serve to identify a general pattern: Mathematical sophistication tends to increase with time, but quite unevenly, leaving some non-marginal areas in their essentially non-mathematized initial state, while others have undergone radical mathematization.

However sophisticated the mathematics involved, they do not necessarily have a profound effect on the field. First, there may be a "lamppost" effect, when a mathematical technique is developed on its own impetus, perhaps to the point of creating an entire academic field, but without actually furthering the original problematic (as seen, at least, from a limited time perspective). But second, and more importantly, even a successful mathematization may concern a strictly limited area, without bringing consequential changes to the overall landscape. Besides the depth of mathematical methods, it is therefore important to distinguish between research programs that aim at engaging the entire field of cognitive science with mathematics and programs which result in minimal engagement, either due to the shallowness of the mathematics employed, or to the limited scope of the program.

Scientific temperaments vary. To some, grand schemes are suspect and their formulation and examination are basically a waste of time. Scholars of that bend will therefore be inclined to turn their attention to well-defined problem areas where mathematics has a serious potential. Others are loth to abandon the initial ambitions of cognitive science, viz. to produce in the fullness of time an integrated account of mind and brain, with a density of conceptual connections at least comparable to that of biology, if not that of physics. And so, in the face of the increasing fragmentation of cognitive science, they turn to mathematics. Some specific proposals of mathematically-induced unification of the field are on offer. To some of these I now turn, by evoking a few representative theorists.

In his landmark monograph,[24] the late physicist Daniel Amit proposed the most elaborated view of cognitive science as the study of the cooperative properties of the brain tissue, in the tradition initiated by Hopfield, but with a novel concern with neurobiological realism. By applying the know-how of the physicist in deploying statistical mechanics and dynamical systems theory to nature's most complex system, the brain, examined with the utmost care at every level, one can hope, according to Amit, to develop a unified theory that would stand to the brain in roughly the same relation as state-of-the-art mathematical physics to (non-biological) natural systems, by establishing systematic links between the various levels of organization. Cognitive science would thus be unified, though not reduced, under the banner of a highly sophisticated mathematical physics.

Paul Smolensky, also trained as a physicist, went on to become the most articulate and powerful theorist of the PDP school,[25] to which he contributed early

24 Daniel J. Amit, *op. cit.*

25 Paul Smolensky, "On the Proper Treatment of Connectionism", in: *The Behavioral and Brain Sciences* 11, 1988, pp. 1–23.

on a unifying framework which he called "harmony theory".[26] Cognitive processes, in a wide (and ever widening) spectrum of cases, consist in attempting to honor a (usually large) number of "soft constraints". It is seldom possible to honor them all, so that the desired outcome is a state where the sum total of violations is minimal: a system that reaches such a state has achieved the highest possible 'harmony'. Now what turns this idea from metaphor to theoretical principle is the mathematics (linear algebra, dynamical systems, probability theory) that shows that, under suitable conditions, a feed-forward multi-layered network can actually achieve a harmony maximum. Characteristically, Smolensky did not rest content with this perspective, which failed to connect with the classical principles and concepts of "classical" or "symbolic" cognitive science. He now regards the central challenge to be to precisely characterize the kind of abstraction that bridges the biophysical properties of the brain to the computational properties of mental representations and knowledge – in short, to the *mind*, and he has taken up the challenge in the area of linguistics.[27] His mental representations are to be understood as abstract theoretical constructs that must be characterized precisely through formal systems developed using the methods of mathematics. Thus, he writes, "The ultimate goal of my work is to help usher cognitive science through a fundamental transition into a truly mathematical discipline."[28]

Methodology, according to Smolensky, has been the great weakness of cognitive science, causing a sterile battle of "isms", speculative theses regarding the true nature of cognition. Between the "ism" level and the "model" level (highly specialized accounts of lab-generated data on very specific behaviors) there lacks what he calls the level of *general theory*, which according to him is "largely missing because sophisticated use of mathematics is required" much of which remains to be created *by adequately trained cognitive scientists*: co-evolution again, as the mathematics that cognitive science requires to come of age is itself yet in limbo. Now why exactly, one may ask, would mathematics be the means to reach the prescribed end? Smolensky's answer can be broken down in two components. First, formalization is indispensable to regiment and justify the use of abstractions (so as not to smuggle in occult properties in the guise of theoretic entities), and convincing formalizations must yield accounts of complex phenomena from small number of principles governing a small number of variables. Mathematics is the only known discipline that can achieve this. Second, cognitive science presents a special challenge, which is to bridge the gap between the essentially continuous physical substratum and the discrete manifestations at the mental level: again, only the mathematics of emergence deployed in nonlinear physics are known to

26 Paul Smolensky, "Information Processing in Dynamical Systems: Foundations of Harmony Theory", in: David E. Rumelhart and James L. McClelland (Eds.), *op. cit.*

27 Paul Smolensky and Géraldine Legendre, *The Harmonic Mind: From Neural Computation to Optimality-Theoretic Grammar Volume I: Cognitive Architecture; Volume II: Linguistic and Philosophical Implications*. Cambridge, MA: MIT Press 2006.

28 Personal communication.

achieve this, for the fairly simple systems studied thus far – and a similar bridge awaits to be constructed between the physical brain and the mind.

Other strong programs are advocated by authors such as Jean Petitot, Scott Kelso, Chris Eliasmith, or Mark Bickhard.[29] Due to space constraints, no attempt will be made to give the reader so much as a flavor of their respective proposals. Petitot's main theoretical sources are dynamical systems theory, nonlinear physics (emergence in disordered systems, phase transitions), differential geometry, on the one hand, and on the other, strikingly, Husserlian phenomenology turned, so to speak, on its head, and thus "naturalized" by virtue of the new mathematical physics which Husserl could not fathom. Kelso takes his inspiration also from dynamical systems and more particularly from Hermann Haken's theory of self-organized nonequilibrium phase transitions. Eliasmith sees control theory, a branch of theoretical computer science, as providing a unifying framework for the necessarily pluralistic theories of the mind and brain. Bickhard develops an approach of his own, "interactivism", which aims at reconfiguring cognitive science by way of rethinking the naturally emerging high-level properties of living organisms that give rise to representations.

Most of these programs, next to several others, are often grouped under the general label of "dynamicism", and blanket arguments are proffered in favor of what is presented as a shared approach. For example, R. D. Beer claims that "By supplying a common language for cognition, for the neurophysiological processes that support it, for non-cognitive human behavior, and for the adaptive behavior of simpler animals, a dynamical approach holds the promise of providing a unified theoretical framework for cognitive science, as well as an understanding of the emergence of cognition in development and evolution."[30] The trouble with such

29 J. A. Scott Kelso, *Dynamic Patterns.* Cambridge, MA: MIT Press 1997; Chris Eliasmith and Charles H. Anderson, *Neural Engineering: Computation, representation and dynamics in neurobiological systems.* Cambridge, MA: MIT Press 2003; Jean Petitot, Francisco Varela, Bernard Pachoud and Jean-Michel Roy (Eds.), *Naturalizing Phenomenology.* Stanford: Stanford University Press 1999; M.H. Bickhard, "The Biological Foundations of Cognitive Science", in: *New Ideas in Psychology* 27, 1, 2009, pp. 75-84. Other programs originate in AI (the new wave of so-called "Artificial General Intelligence", see Ben Goertzel and Cassio Pennachin, *Artificial General Intelligence*, 1st ed., New York: Springer 2007 ; Ben Goertzel, *The Hidden Pattern. A Patternist Philosophy of Mind*. Florida: BrownWalker Press 2006, or robotics (Rodney Brooks, *Cambrian Intelligence: The Early History of the New AI.* Cambridge, MA: MIT Press 1999; Patti Maes, *Designing Autonomous Agents*. Cambridge, MA: MIT Press 1990.

30 Randall D. Beer, "Dynamical Approaches in Cognitive Science", in: *Trends in Cognitive Sciences* 4, 3, 2000, pp. 91-99. It is generally accepted that dynamicism came into existence as a self-aware and visible orientation within cognitive science with the publication of two collections: Robert Port and Tim van Gelder (Eds.), Mind as Motion: Explorations in the Dynamics of Cognition. Cambridge, MA : MIT Press 1995; Esther Thelen and Linda B. Smith (Eds.), *A Dynamic System Approach to the Development of Cognition and Action.* Cambridge, MA: MIT Press 1996; see also Lawrence M. Ward,

global views is that they lead to a battle of "isms", as Smolensky has argued, a battle that no side can win, while true theoretical progress lies in unearthing the scientific substance of the initial slogans. Better then, perhaps, to examine the various programs and judge them on their own merits rather than on their adherence to overly general principles.

5. Strong Programs, Conceptual Reform and the Co-evolution of Cognitive Science and Mathematics *vs.* Pluralism and the Toolbox Philosophy

Still, it is a striking and crucial fact that all of these strong programs aim at putting order into chaos by virtue of bringing cognition under the jurisdiction of mathematics. And here lie three seemingly major difficulties.

First, by their very plurality, they add to the buzzing, booming confusion of a field that they claim to be desperately in need of regimenting. Second, what their proponents are counting on to make this happen, viz. mathematics, cannot yet deliver: the requisite mathematical tools do not exist at the present pioneering stage. Third, of all the specialized languages and disciplines of science, mathematics is the most impenetrable not only to the practitioners of the human and social sciences, philosophers included, but to many biologists and even computer scientists, who are, on the side of the natural sciences, the ones with the strongest ties to cognitive science. It would seem then that most scientists have at best hands-on knowledge of cognition, or of mathematics, but not both, while both are claimed to be necessary if cognitive science is ever going to attain maturity.

The first problem can be counted on being overcome by the mere passage of time. Cognitive science is at a stage where it suffers from an acute case of the "toothbrush problem" – every major figure in the field with a general theory wants to use his own theory, and nobody else's, but the reasonable hope is that this won't last forever, and more particularly that mathematical models will gain wider acceptance and accelerate convergence.

The second problem is more interesting. One lesson to be gleaned from the most casual inspection of research programs such as those just mentioned is that no program for a fully mathematized cognitive science can succeed without a worked-out program for conceptual reform: just like the founders of the field, tomorrow's architects must provide a structural hypothesis, or, in Newell and Simon's terms,[31] a "law of qualitative structure" regarding the ontology of cognition, together with a unifying methodology. This is of course in line with more mature disciplines such as physics and (molecular) biology. Mathematization invariably

Dynamical Cognitive Science. Cambridge, MA: MIT Press 2001.

31 Allen Newell and Herbert A. Simon, "Computer Science as Empirical Inquiry: Symbols and Search", in: *Communications of the ACM* 19, 3, 1976, pp. 113-126.

goes hand in hand with a set of principles, which are part ontological and part epistemic, the latter regulating the necessary abstractions. The principles, in turn, provide traction only insofar as they make the phenomena accessible to mathematics. And, with rare exceptions, the specific mathematical tools must be developed in tandem with the principles (a point forcefully made, in particular, by Petitot and Smolensky, but also included or implied in just about all the detailed proclamations of new paradigms in cognitive science). This situation therefore calls for conceptual reform driven by co-evolution of cognitive science and mathematics.

The third problem is more vexing, and it is not restricted to cognitive science. I can think of two optimistic and one pessimistic responses. First, we can hope (like Smolensky) that a new generation of mathematically savvy cognitive scientists is now emerging from a few pioneering graduate programs, who will be the moving force of cognitive science in the coming years and decades. Second, we can imagine a situation of distributed scientific competence, where the sophisticated mathematics lies in one group of brains, the advanced cognitive science in another group, without there being many brains, or any for that matter, in both groups. Third, and this is the less sanguine view, we can imagine a future where the two orientations remain at an increasing, rather than diminishing, distance from one another. Mathematical cognitive science would evolve into a separate field, with or without occupying center stage: economics and biology are perhaps examples of each scenario.

Yet we should not forget that overarching methodologies stand at one end of a continuum, whose other pole reflects a pure “hands on” philosophy, one which recommends context-sensitive, case-by-case model construction, and sometimes evokes evolutionary theory and the modularity thesis to bolster the case of tinkering as the proper method in cognitive science. This stance countenances a thorough-going pluralism, with at least three dimensions along which models can vary: the level of aggregation, or level of reality dimension, from (say) the synaptic cleft to consciousness or culture; the genus of models (what counts as a model), as determined by the basic science and the methodology; and finally the task domain, from (say) navigation to chess playing, from face recognition to economic behavior, and so forth. The mathematics provides a toolbox to the working cognitive scientist, who constructs models of systems whose function is known or hypothesized, and whose neural realization is sought. The extent to which this plurality of models can be brought under a unifying scheme, and the importance of mathematics in that scheme, remain to be determined.

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